

The Minimal Controllability Problem

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Abstract

We consider the problem of finding the sparsest input vector which makes a given linear system controllable. Specifically, given a matrix $A \in \mathbb{R}^{n \times n}$ define \mathcal{B} be the set of vectors b in \mathbb{R}^n which make the linear system $\dot{x} = Ax + bu$ controllable. We show that the problem of finding the sparsest vector in \mathcal{B} is NP-hard, and even approximating the number of nonzero entries of the sparsest vector in \mathcal{B} within a multiplicative factor of $c \log n$ is NP-hard for some positive c . Moreover, this remains the case even when the matrix A is symmetric. On the positive side, we show that it is possible to find, in polynomial time, a vector $b \in \mathcal{B}$ with at most $c' \log n$ as many nonzero entries as the sparsest vector in \mathcal{B} for some positive c' . This is achieved by a simple greedy heuristic which sequentially adds nonzero entries to b to maximize the rank increase of the controllability matrix.

1 Introduction

This paper considers the problem of controlling large-scale systems where it is impractical or impossible to affect more than a small number of variables. We are motivated by the recent emergence of control-theoretic ideas in the analysis of biological circuits [14], biochemical reaction networks [11], and systems biology [21], where systems of many reactants often need to be driven to desirable states with inputs which influence only a few key variables.

While controllability aspects of linear and nonlinear systems have been amply studied since the introduction of the concept by Kalman [8], it appears that with the exception of a few recent papers little attention has been paid to the interaction of controllability with sparsity. There has been much recent interest in the control of large networks which have a nearly unlimited number of interacting parts - from the aforementioned biological systems, to the smart grid [3, 27], to traffic systems [4, 5] - leading to the emergence of a budding theory of control over networks (see the surveys [15, 17] and the references therein). The enormous size of these systems makes it costly to affect them with inputs which feed into a nontrivial fraction of the states, naturally motivating the study of sparse control strategies.

We will be considering here the continuous-time linear time-invariant system

$$\dot{x} = Ax + bu \tag{1}$$

We will assume that the matrix $A \in \mathbb{R}^{n \times n}$ is known to us and we would like to find a vector b with fewest nonzero entries so that the system of Eq. (1) is controllable. We will refer to this henceforth as the minimal controllability problem. Intuitively, we would like to design an open-loop

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or feedback controller for the system $\dot{x} = Ax$, which may be thought of as a large-scale system which we can afford to influence through an input in only a few places.

Our work follows a number of recent papers on the controllability properties of networks [2, 6, 7, 9–13, 16, 18–20, 24, 25] with similar motivations. These papers sought to find connections between the combinatorial properties of graphs corresponding to certain structured matrices A and vectors b and the controllability of Eq. (1). It is impossible to give a brief overview of this considerable literature, but we refer the interested reader to the recent survey [6]. We note that the existence of easily optimizable necessary and sufficient combinatorial conditions for controllability would, of course, lead to a solution of the minimal controllability problem we have proposed here; however, to date it is only for matrices A corresponding to influence processes on simple graphs like paths and cycles that such conditions have been found [2, 18].

We mention especially the recent papers [10, 12], which have the a similar starting point as the present paper. It is observed in these papers that a brute-force approach to finding the sparsest vector b which makes Eq. (1) controllable might involve testing controllability with all of the $2^n - 1$ possible sparsity patterns, which is computationally prohibitive. This motivates the authors of [10, 11] to consider instead the problem of finding the sparsest vector which only makes Eq. (1) structurally controllable.

Our paper has two main results. The first of them rigorously confirms that intractability of the minimal controllability problem, and shows that this problem is even intractable to approximate. Let us adopt the notation \mathcal{B} for the set of vectors in \mathbb{R}^n which make Eq. (1) controllable. We then have the following theorem.

Theorem 1.1. *Approximating the number of nonzero entries of the sparsest vector in \mathcal{B} within a multiplicative factor of $c \log n$ is NP-hard for some absolute constant[□] $c > 0$; moreover, this remains the case even under the additional assumption that the matrix A is symmetric.*

Thus not only is it impossible to compute the sparsest b unless $P = NP$, it is NP-hard to even approximate the number of entries in this vector within a multiplicative logarithmic factor. This intractability result is discouraging and arguably motivates the approach taken by [10, 11] which sought instead alternative problem formulations which are tractable.

However, we next argue that in many cases it makes sense to stick with the minimal controllability framework. Indeed, from the perspective of designing scalable control laws, a vector $b \in \mathcal{B}$ which is reasonably sparse will do. One would like to avoid the worst-case scenario where b has roughly as many nonzero entries as the dimension n , which may be enormous. Thus a natural approach is to consider relaxing the requirement of finding the sparsest $b \in \mathcal{B}$ to merely finding a reasonably sparse $b \in \mathcal{B}$.

Unfortunately, unless $P = NP$, the above theorem rules out the possibility of finding a vector $b \in \mathcal{B}$ with $c \log n$ as many times nonzero entries as the sparsest vector in \mathcal{B} . However, since $\log n$ scales quite gracefully with n , it may suffice to find a vector $b \in \mathcal{B}$ whose sparsity matches this barrier. Our next theorem shows that this is possible.

Theorem 1.2. *There exists an algorithm whose running time is polynomial in n and the number of bits in the entries of A which, under the assumption \mathcal{B} is nonempty, returns a vector in \mathcal{B} whose number of nonzero entries is at most $c' \log n$ times the sparsest vector in \mathcal{B} , for some absolute constant $c' > 0$.*

[□] An absolute constant is a constant which does not depend on any of the problem parameters, i.e., in this case it does not depend on A or n .

As we will show, the algorithm which achieves this is a simple greedy heuristic which sequentially adds entries to \mathbf{b} to maximize the rank increase of the controllability matrix. Moreover, our simulation results on matrices \mathbf{A} deriving from directed Erdos-Renyi random graphs show that this algorithm usually finds a vector in \mathcal{B} with only a very small number of nonzero entries.

The rest of this paper is dedicated to the proof of these two theorems. Section 2 contains the proof of Theorem 1.1 and a few of its natural corollaries. Section 3 contains the proof of Theorem 1.2. We report on the results of a simulation in Section 4 and some conclusions are drawn in Section 5.

2 Intractability results for minimal controllability problems

This section is dedicated to the proof of Theorem 1.1 and its consequences. We will break the proof of this theorem into two parts for simplicity: first we will show that the minimal controllability problem is NP-hard for a general matrix \mathbf{A} , and then we will argue that the proof can be extended to symmetric matrices.

The proof itself proceeds by reduction to the hitting set problem, defined next.

Definition 2.1. *Given a collection \mathcal{C} of subsets of $\{1, \dots, m\}$, the minimum hitting set problem asks for a set of smallest cardinality that has nonempty intersection with each set in \mathcal{C} .*

The minimum hitting set problem is NP-hard and moreover is NP-hard to find a set whose cardinality is within a factor of $c \log n$ of the optimal set, for some $c > 0$ (see [1, 22] for this hardness result for the set cover problem, easily seen to be equivalent). Moreover, it is easy to see that we can make a few natural assumptions while preserving the NP-hardness of the hitting set problem: we can assume that each set in \mathcal{C} is nonempty and we can assume that every element appears in at least one set. We will argue that hitting set may be encoded in minimal controllability, so that the latter must be NP-hard as well. We next give an informal overview of our argument.

The argument is an application of the PBH test for controllability which tells us that to make Eq. (1) controllable, it is necessary and sufficient to choose \mathbf{b} that is not orthogonal to all the left-eigenvectors of the matrix \mathbf{A} (a proof may be found in textbooks on linear systems, e.g., [26]). It is easy to see that if \mathbf{A} does not have any repeated eigenvalues, this is possible if and only if the support of \mathbf{b} intersects with the support of every left-eigenvector of \mathbf{A} . Defining \mathcal{C} to be the collection of supports of the eigenvectors, this turns into a minimum hitting set problem.

However, the resulting hitting set problem has some structure, so that we cannot conclude that minimal controllability is NP-hard just yet; for example, clearly the sets in the collection \mathcal{C} defined in the previous paragraph are the supports of the columns of an invertible matrix, so they cannot be arbitrarily chosen. In the case where \mathbf{A} is symmetric and its eigenvectors orthogonal, there is more structure still. The challenge is therefore to encode arbitrary hitting set problems within this structure.

We now proceed to the formal argument. First, however, we introduce some notation which we will use throughout the remainder of this paper. We will say that a matrix or a vector is k -sparse if it has at most k nonzero entries, and we will say that the matrix \mathbf{A} is k -controllable if there exists a vector $\mathbf{b} \in \mathcal{B}$ which is k -sparse which makes Eq. (1) controllable. We will sometimes say that a vector \mathbf{b} makes \mathbf{A} controllable, meaning that Eq. (1) is controllable with this particular \mathbf{A}, \mathbf{b} . We will use \mathbf{I}_k to denote the $k \times k$ identity matrix, $\mathbf{0}_{k \times l}$ denote the $k \times l$ zero matrix, and $\mathbf{e}_{k \times l}$ to mean the $k \times l$ all-ones matrix. Finally, \mathbf{e}_i will denote the i 'th basis vector.

Proof of Theorem 1.1, First part. Given a collection \mathcal{C} of p nonempty subsets of $\{1, \dots, m\}$ such that every element appears in at least one set, we define the incidence matrix $C \in \mathbb{R}^{p \times m}$ where we set $C_{ij} = 1$ if the i 'th set contains the element j , and zero otherwise. We then define the matrix V in $\mathbb{R}^{(p+m+1) \times (p+m+1)}$ as

$$V = \begin{pmatrix} 2\mathbf{I}_m & \mathbf{0}_{m \times p} & \mathbf{e}_{m \times 1} \\ C & (m+1)\mathbf{I}_p & \mathbf{0}_{p \times 1} \\ \mathbf{0}_{1 \times m} & \mathbf{0}_{1 \times p} & 1 \end{pmatrix}$$

By construction, V is strictly diagonally dominant, and therefore invertible. We set

$$A(\mathcal{C}) = V^{-1} \text{diag}(1, \dots, m+p+1)V$$

Note that $A(\mathcal{C})$ has distinct eigenvalues and its left-eigenvectors are the rows of V . Moreover A may be constructed in polynomial-time from the collection \mathcal{C} . Indeed, since every element appears in at least one set, we have that it takes at least $\max(m, p)$ bits to describe \mathcal{C} , and inverting V and performing the necessary multiplications to construct A can be done in polynomial time in m, p [23].

We claim that \mathcal{C} has a hitting set with cardinality k if and only if $A(\mathcal{C})$ is $k+1$ -controllable.

Indeed, suppose that a hitting set S' of cardinality k exists for \mathcal{C} . Set $b_i = 1$ for all $i \in S'$ and $b_{m+k+1} = 1$; set all other entries of b to zero. By construction, b has positive inner product with every row of V . The PBH test then implies that the system of Eq. (1) with this b is controllable.

Conversely, suppose the system is controllable with a $k+1$ -sparse vector b . Once again we appeal to the PBH condition, which gives that b is not orthogonal to any row of V . Since V is nonnegative, we may take the absolute value of every entry of b : the PBH test implies this operation preserves the controllability of Eq. (1). Moreover, the PBH condition for $A(\mathcal{C})$ and nonnegative b is that for any row i of V , there is some index j such that both V_{ij} and b_j are positive. We will refer to this as the intersection property.

By considering the last row of V , we see that the intersection property immediately implies that b_{m+k+1} is positive. We now argue that we can find a $k+1$ -sparse vector b whose support is contained in $\{1, \dots, m\} \cup \{m+p+1\}$ such that the system of Eq. (1) is controllable. Indeed, if $b_i > 0$ for some $i \notin \{1, \dots, m\} \cup \{m+p+1\}$ then we modify b by setting first $b_i = 0$ and then $b_l = 1$ for any index $l \in \{1, \dots, m\}$ such that $V_{il} > 0$, i.e. any l belonging to the i 'th set of \mathcal{C} . In so doing, we preserve the controllability of Eq. (1) since, by construction, for $i \notin \{1, \dots, m\} \cup \{m+p+1\}$ no row besides i has i 'th entry positive, so the intersection condition still holds. Moreover, the vector b certainly remains $k+1$ -sparse.

Proceeding in this way for every $i \notin \{1, \dots, m\} \cup \{m+p+1\}$ we finally get a $k+1$ -sparse vector b whose support is contained in $\{1, \dots, m\} \cup \{m+p+1\}$ and which makes Eq. (1) controllable. The intersection property then implies that for each $i = 1, \dots, p$ there is some index j such that $b_j > 0$ and the $m+i$ 'th row of V is positive on the j 'th entry. By construction, this means that $\text{supp}(b) \cap \{1, \dots, m\}$ is a hitting set for \mathcal{C} . Since b is $k+1$ -sparse and $b_{m+p+1} > 0$, this hitting set has cardinality at most k .

This concludes the proof of the italicized claim above that the solution of the minimal controllability problem is one more than the size of the smallest hitting set for \mathcal{C} . This implies that the intractability guarantees for hitting set extend to minimal controllability. \square

This proves Theorem 1.1 without the additional assumption that A is a symmetric matrix. Before proceeding to describe the proof in the symmetric case, we illustrate the construction with a simple example.

Example: Suppose

$$\mathcal{C} = \{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 2, 3\}.$$

Here $m = 3$ and $p = 4$. The smallest hitting set is of size 2, but there is no hitting set of size 1. The incidence matrix is

$$C = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

and

$$V = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 4 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 4 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and $A = V^{-1}\text{diag}(1, 2, 3, 4, 5, 6, 7, 8)V$ is

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & -7/2 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & -3 \\ 0 & 0 & 3 & 0 & 0 & 0 & 0 & -5/2 \\ 3/4 & 1/2 & 0 & 4 & 0 & 0 & 0 & 13/7 \\ 0 & 3/4 & 1/2 & 0 & 5 & 0 & 0 & 11/8 \\ 5/4 & 0 & 3/4 & 0 & 0 & 6 & 0 & 3/2 \\ 3/2 & 5/4 & 1 & 0 & 0 & 0 & 7 & 9/4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 \end{pmatrix}$$

Direct computation shows that choosing $b = (1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1)^T$ makes the system controllable and our proof above implies that no 2-sparse vector b makes it controllable.

We now proceed with the proof of Theorem 1.1, which remains to be shown in the case the matrix is symmetric. For this we will need the following lemma, which is a minor modification of known facts about the Gram-Schmidt process.

Lemma 2.2. *Given a collection of k orthogonal vectors v_1, \dots, v_k all in \mathbf{e}_1^\perp in \mathbb{R}^n , whose entries are rational with bit-size B , it is possible to find in polynomial time in n, B vectors v_{k+1}, \dots, v_n such that: (i) v_1, \dots, v_n is an orthogonal collection of vectors (ii) the first component of each of the vectors v_{k+1}, \dots, v_n is nonzero.*

Proof. Set $v_{k+1} = \mathbf{e}_1$ and use Gram-Schmidt to find v_{k+2}, \dots, v_n so that v_1, \dots, v_n is an orthogonal basis for \mathbb{R}^n . Now let \mathcal{S} be the set of those vectors among v_{k+1}, \dots, v_n which have zero first entry; clearly we have $\mathcal{S} = \{k+2, k+3, \dots, n\}$ since v_{k+1} has first entry of 1 and zeros in all other entries and v_{k+2}, \dots, v_n are orthogonal to v_{k+1} .

Now given a collection v_1, \dots, v_n such that v_{k+1} has first entry of 1, we describe an operation that decreases the size of \mathcal{S} . Pick any $v_l \in \mathcal{S}$ and update as

$$\begin{aligned} v_l &\leftarrow cv_l + v_{k+1} \\ v_{k+1} &\leftarrow -v_l + v_{k+1} \end{aligned}$$

where $c = \|v_{k+1}\|_2^2 / \|v_l\|_2^2$. Observe that the collection v_1, \dots, v_n remains orthogonal after this update and that the new vectors v_{k+1}, v_l both have 1 as their first entry.

As long as there remains a vector in \mathcal{S} , the above operation decreases the cardinality of \mathcal{S} by 1. It follows that we can make \mathcal{S} empty, which is what we needed to show. It is easy to see that the procedure takes polynomial-time in n, B . \square

We now complete the proof of Theorem 1.1.

Proof of Theorem 1.1, Second part. Let us introduce the notation $\text{mc}(X)$ to mean the number of nonzero entries in a solution b of the minimal controllability problem with the matrix X . We will next construct, in polynomial time, a symmetric matrix \hat{A} and prove that it satisfies the inequality $\text{mc}(\hat{A})/\text{mc}(A) \in [1/3, 2]$. Once this has been established we will then have that since approximating $\text{mc}(A)$ to within a multiplicative factor of $c \log n$ is NP-hard for some $c > 0$, the same holds for $\text{mc}(\hat{A})$.

We will construct \hat{A} as follows: we will take V and add $\binom{m+p+1}{2} + 1$ columns to it; then we will add a corresponding number of rows to make the resulting matrix square. We will call the resulting matrix \hat{V} ; we have that $\hat{V} \in \mathbb{R}^{r \times r}$ where $r = m + p + 2 + \binom{m+p+1}{2}$.

We now describe how we fill in the extra entries of \hat{V} . Let us index the first $\binom{m+p+1}{2}$ additional columns by $\{i, j\}$ for $i, j = 1, \dots, m + p + 1$. If the i 'th and j 'th rows of V have nonzero inner product, we set the one of $\hat{V}_{i, \{i, j\}}, \hat{V}_{j, \{i, j\}}$ to 1 and the other to the negative of the inner product with the i 'th and j 'th rows of V ; else, we set both of them to zero. All other additional entries in the first $m + p + 1$ rows of \hat{V} will be set to zero. Note that by construction the first $m + p + 1$ rows of \hat{V} are orthogonal.

As for the extra rows of \hat{V} , we will fill them in using Lemma 2.2 to be orthogonal to the existing rows of V and have a nonzero entry in the r 'th coordinate. By construction the rows of \hat{V} are orthogonal. Finally, $\hat{A} = \hat{V}^{-1} \text{diag}(1, 2, \dots, r) \hat{V}$. Note that A is symmetric and the left-eigenvectors of \hat{A} are the rows of \hat{V} .

Now that we have constructed \hat{A} , we turn to the proof of the assertion that $\text{mc}(\hat{A})/\text{mc}(A) \in [1/3, 2]$. Indeed, suppose A is k -controllable, i.e., there exists a k -sparse vector $b \in \mathbb{R}^{m+p+1}$ which makes Eq. (1) controllable. We then define a $k + 1$ -sparse vector $\hat{b} \in \mathbb{R}^r$ by setting its first $m + p + 1$ entries to the entries of b and setting its r 'th entry to a random number generated according to any continuous probability distribution, say a standard Gaussian distribution; the rest of the entries of \hat{b} are zero. By construction, rows $1, \dots, m + p + 1$ of \hat{V} are not orthogonal to \hat{b} ; and the probability that any other row is orthogonal to \hat{b} is zero. We have thus generated a $k + 1$ -sparse vector \hat{b} which makes \hat{A} controllable with probability 1. We thus conclude that there exists a $k + 1$ sparse vector which makes \hat{A} controllable, for if such a vector did not exist, the probability that \hat{b} makes \hat{A} controllable would be zero, not one. Finally, noting that $(k + 1)/k \leq 2$, we conclude that $\text{mc}(\hat{A}) \leq 2\text{mc}(A)$.

For the other direction, suppose now that \hat{A} is controllable with a k -sparse vector \hat{b} . We argue that there exists a $k + 1$ sparse vector b' making \hat{A} controllable which has the property that $b'_j = 0$ for all $j \notin \{1, \dots, m + p + 1\} \cup \{r\}$.

Indeed, we now describe how we can manipulate \hat{b} to obtain b' . Let \mathcal{S} be the support of \hat{b} . If some element j not in $\{1, \dots, m + p + 1\} \cup \{r\}$ is in \mathcal{S} , we remove it by setting $b_j = 0$; now observe that at most two of the first $m + p + 1$ rows of \hat{V} have a nonzero entry in the j 'th place; we add two elements in $\{1, \dots, m + p + 1\}$ to \mathcal{S} , one from the support of each of those two rows within $\{1, \dots, m + p + 1\}$, by setting the corresponding entries of b to a continuous positive random

variable. Specifically, suppose that $j \in \mathcal{S}$ for some $j \notin \{1, \dots, m+p+1\} \cup \{r\}$, and $\widehat{V}_{i_1, j} > 0$ and $\widehat{V}_{i_2, j} > 0$ for $i_1, i_2 \in \{1, \dots, m+p+1\}$ are the two rows among the first $m+p+1$ with a nonzero coordinate in the j 'th spot; we then pick any index $k_1 \in \{1, \dots, m+p+1\}$ in the support of the i_1 'st row of \widehat{V} and index $k_2 \in \{1, \dots, m+p+1\}$ in the support of the i_2 'nd row of \widehat{V} and we set b_{k_1} and b_{k_2} to be, say, independent uniform random variables on $[1, 2]$. Finally, we add r to \mathcal{S} by setting b_r to a positive continuous random variable, to make sure that the rows $m+p+2, \dots, r$ have supports which overlap with the support of b .

Proceeding this way, we end up with a support \mathcal{S} which is at most three times as many elements as it had initially - because each time we remove an index we add at most three indices - but which has no element outside of $\{1, \dots, m+p+1\} \cup \{r\}$. Finally, we note that for any row of \widehat{V} , the probability that it is orthogonal to b' is zero. Thus with probability one, the vector b' makes \widehat{A} controllable. Consequently, we conclude there exists a $3k$ sparse vector that makes \widehat{A} controllable.

It now follows that the vector $b \in \mathbb{R}^{m+p+1}$ obtained by taking the first $m+p+1$ entries of b' makes A controllable. Thus $\text{mc}(A) \leq 3\text{mc}(\widehat{A})$. This concludes the proof. \square

We now illustrate the proof with an example intended to make the construction clearer.

Example: In the interest of keeping the dimensions of the resulting matrices from getting too large, let us take a simpler example than before. Let

$$\mathcal{C} = \{\{1\}, \{1, 2\}\}.$$

Thus $\{1\}$ is a hitting set of size one, but not $\{2\}$. We have that

$$V = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 2 & 0 & 0 \\ 1 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and the initial part of \widehat{V} is:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 2 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}$$

Note that the rows of this matrix are orthogonal by construction. We can now use the procedure of Lemma 2.2 to complete this to a matrix with orthogonal rows with positive final entry. Instead, we do a quick approximation here using the MATLAB “qr” command. Our approximation is based on the observation that running Gram-Schmidt process with vectors $\mathbf{e}_6 + \mathbf{e}_{16}, \mathbf{e}_7 + \mathbf{e}_{16}, \dots, \mathbf{e}_{15} + \mathbf{e}_{16}, \mathbf{e}_{16}$ produces the desired result. The matrix \widehat{V} , as computed by the MATLAB “qr” command to three

digits of precision, is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 2 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0.130 & -0.157 & 0.000 & 0.000 & -0.026 & -0.496 & 0.130 & 0.130 & 0.130 & 0 & -0.157 & -0.157 & 0 & -0.0000 & 0 & -0.783 \\ -0.054 & 0.091 & -0.254 & -0.000 & 0.163 & 0.435 & -0.689 & 0.073 & 0.073 & 0 & 0.091 & 0.091 & 0 & -0.127 & 0 & -0.435 \\ -0.009 & -0.039 & 0.085 & -0.206 & 0.115 & 0.307 & 0.307 & -0.772 & 0.051 & 0 & 0.167 & 0.064 & 0 & 0.146 & 0 & -0.307 \\ 0.111 & 0.084 & 0.075 & 0.056 & -0.255 & 0.270 & 0.270 & 0.270 & -0.667 & 0 & 0.028 & 0.412 & 0 & 0.0094 & 0 & -0.270 \\ -0.104 & 0.071 & 0.063 & 0.047 & -0.017 & 0.030 & 0.030 & 0.030 & 0.030 & -0.985 & 0.024 & -0.047 & 0 & 0.008 & 0 & -0.030 \\ -0.307 & 0.242 & 0.073 & -0.265 & 0.274 & 0.008 & 0.008 & 0.008 & 0.008 & 0.0083 & -0.773 & 0.265 & 0 & 0.169 & 0 & -0.008 \\ -0.279 & 0.702 & 0.163 & -0.025 & -0.281 & 0.140 & 0.140 & 0.140 & 0.140 & 0.140 & 0.140 & -0.421 & 0 & 0.094 & 0 & -0.140 \\ -0.060 & -0.030 & 0.039 & 0.052 & 0.020 & 0.010 & 0.010 & 0.010 & 0.010 & 0.010 & 0.010 & 0.010 & -0.995 & -0.007 & 0 & -0.010 \\ -0.120 & -0.060 & 0.477 & -0.297 & 0.040 & 0.020 & 0.020 & 0.020 & 0.020 & 0.020 & 0.020 & 0.020 & 0.020 & -0.813 & 0 & -0.020 \\ -0.057 & -0.029 & 0.029 & 0.057 & 0.019 & 0.010 & 0.010 & 0.010 & 0.010 & 0.010 & 0.010 & 0.010 & 0.010 & 0.010 & -0.995 & -0.010 \\ 0.586 & 0.293 & -0.293 & -0.586 & -0.195 & -0.098 & -0.098 & -0.098 & -0.098 & -0.098 & -0.098 & -0.098 & -0.098 & -0.098 & -0.098 & 0.098 \end{pmatrix}$$

Because this is a matrix with orthogonal rows (up to a certain degree of precision - if we used the procedure of Lemma 2.2, we would have a matrix with exactly orthogonal rows, but because we used MATLAB's built in operations, this matrix has orthogonal rows up to some round-off error), the matrix \hat{A} is symmetric. As we have argued above, the vector $b = (1 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ X)$ where X is a continuous random variable makes \hat{A} controllable with probability one. A direct computation in fact reveals that in this case even $b = (1 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0)$ makes the system controllable, so that $\text{mc}(\hat{A})/\text{mc}(A) = 1$ here. Consistent with our proof, the vector $(0 \ 1 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ X)$ does not make \hat{A} controllable because $\{2\}$ is not a hitting set for \mathcal{C} .

We now proceed to state and prove several natural corollaries of Theorem 1.1. The first of these states that the problem remains NP-hard if the vector b is replaced by a matrix with an arbitrary number of columns.

Corollary 2.3. *For any integer $l \geq 1$, the problem of finding a matrix $B \in \mathbb{R}^{n \times l}$ with smallest number of nonzero entries such that the system*

$$\dot{x} = Ax + Bu$$

is controllable is NP-hard. Moreover, even approximating the smallest possible number of nonzero entries in such a B to within a factor of $c \log n$ is NP-hard for some $c > 0$. This remains the case even if the matrix A is assumed to be symmetric.

Proof. Consider the matrix $\hat{A} \in \mathbb{R}^r$ constructed in the proof of Theorem 1.1. We claim that a k -sparse vector $b \in \mathbb{R}^r$ exists that makes Eq. (1) controllable if and only if a k -sparse matrix $B \in \mathbb{R}^{r \times l}$ exists with this property.

One direction is trivial: if a k -sparse b exists, then certainly a k -sparse matrix B exists - take $B = [b \ 0]$.

For the reverse direction, suppose such a k -sparse matrix B exists. Let b be a random vector such that b_i is equal to an i.i.d. standard Gaussian if the i 'th row of B has a nonzero entry, and zero otherwise. Clearly, any vector obtained this way is k -sparse. Moreover, the PBH condition applied to (A, B) implies that for every row v_i of \hat{V} , there exists at least one column of B which has a nonzero entry in the same index at which v_i has a nonzero entry. It follows that for every v_i , there is an index l such that the l 'th entry of v_i is nonzero and b_l is drawn from a standard Gaussian. This implies that v_i is orthogonal to b with probability zero, and since this is true for all v_i we conclude that the random vector b makes \hat{A} controllable with probability 1. It follows there exists a k -sparse vector b which makes \hat{A} controllable. \square

A variation of the minimal controllability involves finding matrices B and C that make the system both controllable and observable, with as few nonzero entries as possible. Unfortunately, the next corollary proves an intractability result in this scenario as well.

Corollary 2.4. *For any $l_1 \geq 1, l_2 \geq 1$ finding matrices $B \in \mathbb{R}^{n \times l_1}, C \in \mathbb{R}^{l_2 \times n}$ with the smallest total number of nonzero entries (in both B and C) such that the system*

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{2}$$

is both controllable and observable is NP-hard.

Proof. Consider the matrix V which was constructed in the proof of Theorem 1.1. We first argue that V^{-1} has the same pattern of nonzero entries as V , with the exception of the final column of V^{-1} , which is nonzero in every entry. We will show this by explicitly computing V^{-1} .

Indeed, for $i = 1, \dots, m$ we have

$$(Vx)_i = 2x_i + x_{m+p+1}$$

and moreover $(Vx)_{p+m+1} = x_{p+m+1}$ so that the i 'th row of V^{-1} is immediate: it has a $1/2$ in the i, i 'th place and a $-1/2$ in the $i, m+p+1$ 'th place, and zeros elsewhere.

Next for $i = m+1, \dots, m+p$, we have

$$(Vx)_i = (m+1)x_i + \sum_{j \in C_i} x_j$$

where C_i is the corresponding set of the collection \mathcal{C} . Consequently, V^{-1} has a $1/(m+1)$ in the i, i 'th spot, a $-1/(2(m+1))$ in the i, j 'th place where $j \in C_i$, and $|C_j|/(2(m+1))$ in the $i, m+p+1$ 'th place; every other entry of the i 'th row is zero. Finally, the last row of V^{-1} is clearly $[0, 0, \dots, 0, 1]$.

This explicit computation shows that V^{-1} has the same pattern of nonzero entries as V , with the exception of the final column of V^{-1} which is nonzero in every entry. Now for the system of Eq. (2) to be observable, we need to satisfy the PBH conditions, namely that each column of V^{-1} is not orthogonal to some row of C . By considering columns $m+1, \dots, m+p+1$, we immediately see that C needs to have at least $p+1$ nonzero entries; moreover, it is easy to see that $p+1$ nonzero entries suffice, for example by setting the last $p+1$ entries of the first row of C to 1, and all other entries to zero.

Thus the problem of minimizing the number of nonzero entries in both B and C to make Eq. (2) both observable and controllable reduces to finding the sparsest B to make it controllable. We have already shown that this problem is NP-hard. \square

3 Approximating minimal controllability

We will now consider the problem of controlling a linear system with a vector which does not have too many nonzero entries. Our goal in this section will be to prove Theorem 1.2 showing that we can match the lower bound of Theorem 1.1 by returning a $c' \log n$ approximate solution for the minimal controllability problem of a matrix in $A \in \mathbb{R}^{n \times n}$.

For simplicity, let us adopt the notation $|b|$ for the number of nonzero entries in the vector b . Our first step will be to describe a randomized algorithm for the minimal controllability problem which

assumes the ability to set entries of the vector b to Gaussian random numbers. This is not a plausible assumption: in the real-world, we can generate random bits, rather than infinite-precision random numbers. Nevertheless, this assumption simplifies the proof somewhat and clarifies the main ideas involved. After giving an algorithm for approximating minimal controllability with this unrealistic assumption, we will show how to modify this algorithm to work without this assumption. The final result is a deterministic polynomial-time algorithm for $c' \log n$ -approximate controllability.

We begin with a statement of our randomized algorithm in the box below (we will refer to this algorithm henceforth as *the randomized algorithm*). The algorithm has a very simple structure: given a current vector b it tries to update it by setting one of the zero entries to an independently generated standard Gaussian. It picks the index which maximizes the increase in the rank of the controllability matrix $C(A, b) = (b \quad Ab \quad A^2b \quad \dots \quad A^{n-1}b)$. When it is no longer able to increase the rank of $C(A, b)$ by doing so, it stops.

-
1. Initialize b to the zero vector and set $c^* = 1$.
 2. **While** $c^* > 0$,
 - (a) **For** $j = 1, \dots, n$,
 - i. If $b_j = 0$, set $\tilde{b} = b + X(j)\mathbf{e}_j$ where $X(j)$ is i.i.d. standard normal.
 - ii. Set $c(j) = \text{rank}(C(A, \tilde{b})) - \text{rank}(C(A, b))$.
 - end**
 - (b) Let $j^* \in \arg \max_j c(j)$ and let $c^* = c(j^*)$.
 - (c) If $c^* > 0$, set $b \leftarrow b + X(j^*)\mathbf{e}_{j^*}$.
 - end**
 3. Output b .
-

Our first proposition will show that if \mathcal{B} (recall, \mathcal{B} is the set of vectors which make the system of Eq. (1) controllable) is nonempty, then this algorithm returns a vector in \mathcal{B} with probability 1. Moreover, the number of nonzero entries in a vector returned by this algorithm approximates the sparsest vector in \mathcal{B} by a factor of $c' \log n$, for some $c' > 0$.

Without loss of generality, we will be assuming henceforth that each eigenspace of A is one-dimensional: if some eigenspace of A has dimension larger than 1 then \mathcal{B} is empty and all the results we will prove in this section are vacuously true.

Let us adopt the notation $\lambda_1, \dots, \lambda_k$ for the eigenvalues of A . Let T be a matrix that puts A into Jordan form, i.e., $T^{-1}AT = J$, where J is the Jordan form of A . Due to the assumption that every eigenspace of A is one-dimensional, we have that J is block-diagonal with exactly k blocks.

Without loss of generality, let us assume that the i 'th block is associated with the eigenvalue λ_i . Moreover, let us denote its dimension by d_i , and, finally, let us introduce the notation $t(i, j)$ to mean the $d_1 + d_2 + \dots + d_{i-1} + j$ 'th row of T^{-1} . Observe that for fixed i , the collection $\{t(i, j) \mid j = 1, \dots, d_i\}$ comprise the rows of T^{-1} associated with the i 'th block of J . Naturally, the entire collection

$$\mathcal{T} = \{t(i, j), i = 1, \dots, k, j = 1, \dots, d_i\}$$

is a basis for \mathbb{R}^n . For a vector $v \in \mathbb{R}^n$ we use $v[i]$ to denote the vector in \mathbb{R}^{d_i} whose k 'th entry equals the $d_1 + \dots + d_{i-1} + k$ 'th entry of v . We will say that the vector $t(i, j) \in \mathcal{T}$ is *covered* by b if $\langle t(i, j'), b \rangle \neq 0$ for some $j' \leq d_i$ such that $j' \geq j$; else, we will say that $t(i, j)$ is *uncovered* by b .

Our first lemma of this section provides a combinatorial interpretation of the rank of the controllability matrix in terms of the number of covered $t(i, j)$. While the lemma and its proof are quite standard, we have been unable to find a clean reference in the literature; consequently, we provide a self contained proof here.

Lemma 3.1. *$\text{rank}(C(A, b))$ equals the number of covered $t(i, j) \in \mathcal{T}$.*

Proof. Let U_k denote the upper-shift operator on \mathbb{R}^k and let J_i be the i 'th diagonal block of J so that $J_i = \lambda_i \mathbf{I}_{d_i} + U_{d_i}$. Since $\text{rank}(C(A, b)) = \text{rank}(C(T^{-1}AT, T^{-1}b))$, defining $b_i = (T^{-1}b)[i]$ we have

$$\text{rank}(C(A, b)) = \sum_{i=1}^k \text{rank} \left(\begin{pmatrix} b_i & J_i b_i & J_i^2 b_i & \dots & J_i^{d_i-1} b_i \end{pmatrix} \right)$$

Since the rank is unaffected by column operations, we can rewrite this as

$$\text{rank}(C(A, b)) = \sum_{i=1}^k \text{rank} \left(\begin{pmatrix} b_i & U_{d_i} b_i & U_{d_i}^2 b_i & \dots & U_{d_i}^{d_i-1} b_i \end{pmatrix} \right)$$

But each term in this sum is just the largest index of a nonzero entry of the vector b_i . That is, letting z_i be the largest $j \in \{1, \dots, d_i\}$ such that the j 'th entry of b_i is nonzero, we get

$$\text{rank}(C(A, b)) = \sum_{i=1}^k z_i$$

This is exactly the number of covered elements in \mathcal{T} . □

Our next lemma uses the combinatorial characterization we have just derived to obtain a performance bound for the randomized algorithm we have described.

Lemma 3.2. *Suppose that \mathcal{B} is nonempty and let b' be a vector in \mathcal{B} . Then with probability 1 the randomized algorithm outputs a vector in \mathcal{B} with $O(|b'| \log n)$ nonzero entries.*

Proof. Since $b' \in \mathcal{B}$, we know that every $t(i, j) \in \mathcal{T}$ is covered by b' . Thus, in particular, if \mathcal{F} is any subset of \mathcal{T} , then every $t(i, j) \in \mathcal{F}$ is covered by b' . From this we claim that we can draw the following conclusion: for any $\mathcal{F} \subset \mathcal{T}$ there exist an index $j \in \{1, \dots, n\}$ such that the choice $b = e_j$ covers at least $|\mathcal{F}|/|b'|$ of the vectors in \mathcal{F} .

Indeed, if this were not true, then for every index j the number of $t(i, j)$ such that some $t(i, j')$ with $j \leq j' \leq d_i$ has a nonzero j' 'th entry is strictly fewer than $|\mathcal{F}|/|b'|$. It would then follow that there is at least one $t(i, j)$ such that no $t(i, j'), j \leq j' \leq d_i$ has a nonzero entry at any index where b' is nonzero. But this contradicts every $t(i, j)$ being covered by b' .

Let us consider the l 'th time when the randomized algorithm enters step 2.a. Let $\mathcal{F}(l)$ be the number of uncovered vectors in \mathcal{T} at this stage. We have just shown that there exists some index j such that at least $|\mathcal{F}(l)|/|b'|$ of the vectors in $\mathcal{F}(l)$ have nonzero j 'th entry. With probability 1, all of these vectors will become covered when we put an independent standard Gaussian in j 'th

entry, and also with probability 1, no previously covered vector becomes uncovered. Thus we can infer the following conclusions with probability 1: first

$$|\mathcal{F}(l+1)| \leq |\mathcal{F}(l)| - \frac{|\mathcal{F}(l)|}{|b'|},$$

since the index picked by the algorithm covers at least as many vectors as the index j ; and also since $(1 - \frac{1}{x})^x \leq e^{-1}$ for all $x \geq 1$, we have that $|\mathcal{F}(l)|$ shrinks by a factor of at least e^{-1} every $|b'|$ steps. After $O(|b'| \log n)$ steps, $|\mathcal{F}(l)|$ is strictly below 1, so it must equal zero. This proves the lemma. \square

By choosing b' to be the sparsest vector in \mathcal{B} and applying the lemma we have just proved, we have the approximation guarantee that we seek: the randomized algorithm finds a vector in \mathcal{B} which is an $O(\log n)$ approximation to the sparsest vector.

We next revise our algorithm by removing the assumption that we can generate infinite-precision random numbers. The new algorithm is given in a box below, and we will refer to it henceforth as *the deterministic algorithm*.

-
1. Initialize b to the zero vector and set $c^* = 1$.
 2. **While** $c^* > 0$,
 - (a) **For** $j = 1, \dots, n$,
 - i. If $b_j = 0$, then for $p = 1, \dots, 2n+1$, set $\tilde{b}_{j,p} = b + p\mathbf{e}_j$.
 - ii. Set $c(j, p) = \text{rank}(C(A, \tilde{b}_{j,p})) - \text{rank}(C(A, b))$.
 - end**
 - (b) Let $(j^*, p^*) \in \arg \max_{(j,p)} c(j, p)$ and let $c^* = c(j^*, p^*)$.
 - (c) If $c^* > 0$, set $b \leftarrow b + p^* \mathbf{e}_{j^*}$.
 - end**
 3. Output b .
-

The deterministic algorithm has a simple interpretation. Rather than putting a random entry in the j 'th place and testing the corresponding increase in the rank of the controllability matrix, it instead tries $2n+1$ different numbers to put in the j 'th place, and chooses the one among them with the largest corresponding increase in rank. The main idea is that to obtain the “generic” rank increase associated with a certain index we can either put a random number in the j 'th entry or we can simply test what happens when we try enough distinct values in the j 'th entry.

We are now finally ready to prove Theorem 1.2.

Proof of Theorem 1.2. We will show that when the deterministic and the randomized algorithms enters step 2.b with identical A, b , we will have that $c(j)$ in the randomized algorithms equals $\max_p c(j, p)$ in the deterministic algorithm with probability 1. This implies that with probability 1 the sets $\arg \max_j c(j)$ and $\arg \max_{j,p} c(j, p)$ are identical. We have already shown an $O(\log n)$ approximation to the sparsest vector in \mathcal{B} for the randomized algorithm (which picks an arbitrary

index from $\arg \max_j c(j)$ at each step); it is easy to see that showing the above fact will immediately imply the $O(\log n)$ approximation for the deterministic algorithm as well.

Now we turn to the proof of the claim in the above paragraph. Fix A, b, j , and consider $p_m(t)$ which we define to be the sum of the squares of the determinants of all the $m \times m$ minors of the matrix $C(A, b + te_j)$. Naturally, $p_m(t)$ is a nonnegative polynomial in t of degree at most $2n$. Moreover, the statement $p_m(t) > 0$ is equivalent to the assertion that $C(A, b + te_j)$ has rank at least m . Note that because $p_m(t)$ is a polynomial and therefore has finitely many roots if not identically zero, we have that as long as $p_m(t)$ is not identically zero, $p_m(t)$ evaluated a standard Gaussian is positive with probability 1. Consequently, whenever the randomized algorithm enters step 2.a.ii and computes $c(j)$, we have that with probability 1 this $c(j)$ equals the largest $m \in \{1, \dots, n\}$ such that $p_m(t)$ is not identically zero.

Similarly, consider the deterministic algorithm as it enters step 2.a. Observe that as long as $p_m(t)$ is not identically zero, plugging in one of $1, 2, \dots, 2n + 1$ into $p_m(t)$ will produce a positive number because $p_m(t)$ has degree at most $2n$. As a consequence, $\max_k c(j, k)$ equals the largest m such that $p_m(t)$ is not identically zero. This proves that each $c(j)$ in the randomized algorithm equals $\max_k c(j, k)$ in the deterministic algorithm with probability 1. This concludes the proof of the claim from the first paragraph and consequently concludes the proof. \square

4 A simulation

We briefly report on a result of a MATLAB simulation of our randomized algorithm. We used the MATLAB “randn” command to approximately produce a Gaussian random variable. To obtain the matrices A , we generated a directed Erdos-Renyi random graph where each link was present with a probability of $2 \log n / n$. We constructed the adjacency matrices of these graphs and threw out those which had two eigenvalues which were at most 0.01 apart. The purpose of this was to rule out matrices which had eigenspaces that were more than one dimensional, and due to the imprecise nature of eigenvalue computations with MATLAB, testing whether two eigenvalues were exactly equal did not ensure this. Thus the resulting matrix was controllable with some b .

We then ran the randomized algorithm of the previous section to find a sparse vector making the system controllable. We note that a few minor revisions to the algorithm were necessary due to the ill-conditioning of the controllability matrices. In particular, with A being the adjacency matrix of a directed Erdos-Renyi graph on as few as 20 vertices and b a randomly generated vector, the MATLAB command “rank(ctrb(A,b))” often returns nonsensical results due to the large condition number of the controllability matrix. We thus computed the rank of the controllability matrix by initially diagonalizing A and counting the number of eigenvectors orthogonal to b .

We ran our simulation up to number of nodes $n = 100$, generating 100 random graphs for each n and we found that almost all the matrices could be controlled with a b with just one nonzero entry, and all could be controlled with a b that had two nonzero entries. Specifically, out of the 10,000 random adjacency matrices generated in this way, 9,990 could be controlled with a 1-sparse v and the remaining 10 could be controlled with a 2-sparse b . It appears that the randomized protocol can successfully find very sparse vectors making the system controllable in this randomized scenario.

5 Conclusions

We have shown that it is NP-hard to approximate the minimal controllability factor within a factor of $c \log n$ for some $c > 0$, and we have provided an algorithm which approximates it to a factor of $c' \log n$ for some $c' > 0$. Up to the difference in the constants between c and c' , this resolves the polynomial-time approximability of the minimal controllability problem.

The study of minimal controllability problems is relatively recent and quite a few open questions remain. For example, it would be interesting to understand for which matrices A the minimal controllability question can be solved in polynomial-time. This relates to the combinatorial structure of the eigenvectors of A and will likely require theorems relating the combinatorics of the nonzero entries of A to the combinatorics of the nonzero entries of the matrix of eigenvectors.

It is reasonable to anticipate that real-world networks may have certain “generic” features which considerably simplify the minimal controllability problem. It would therefore be interesting to find algorithms for minimal controllability which always return the correct answers and which “generically” finish in polynomial time. Finally, an understanding of more nuanced features of controllability (i.e., how the energy required to move a network from one state to another depends on its structure) appears to be lacking at the moment.

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